Simple nonunifilar binary word generators

Sarah Marzen

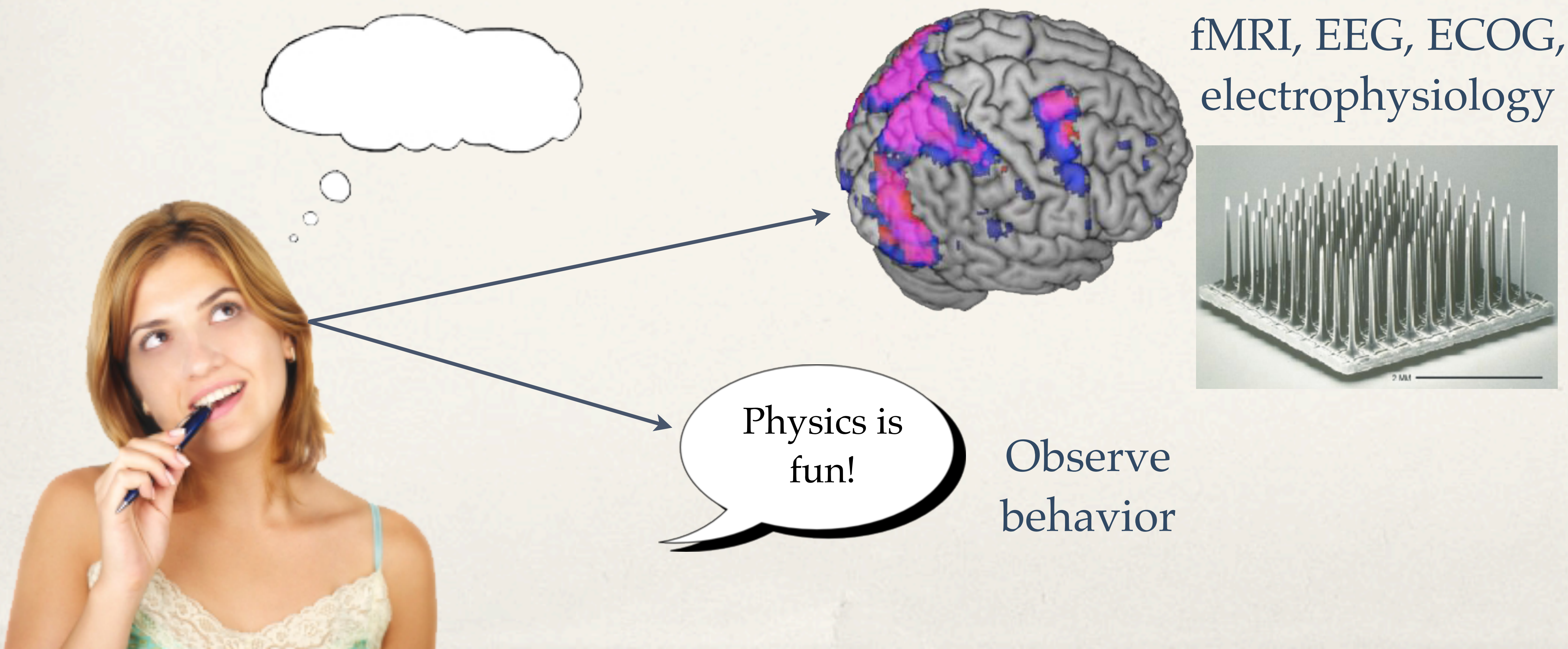
June 1, 2013

Outline

- Motivation
- Results
  - Simple nonunifilar source (SNS)
  - Variation on the SNS
  - A different set of nonunifilar binary word generators (preliminary)
- Future work

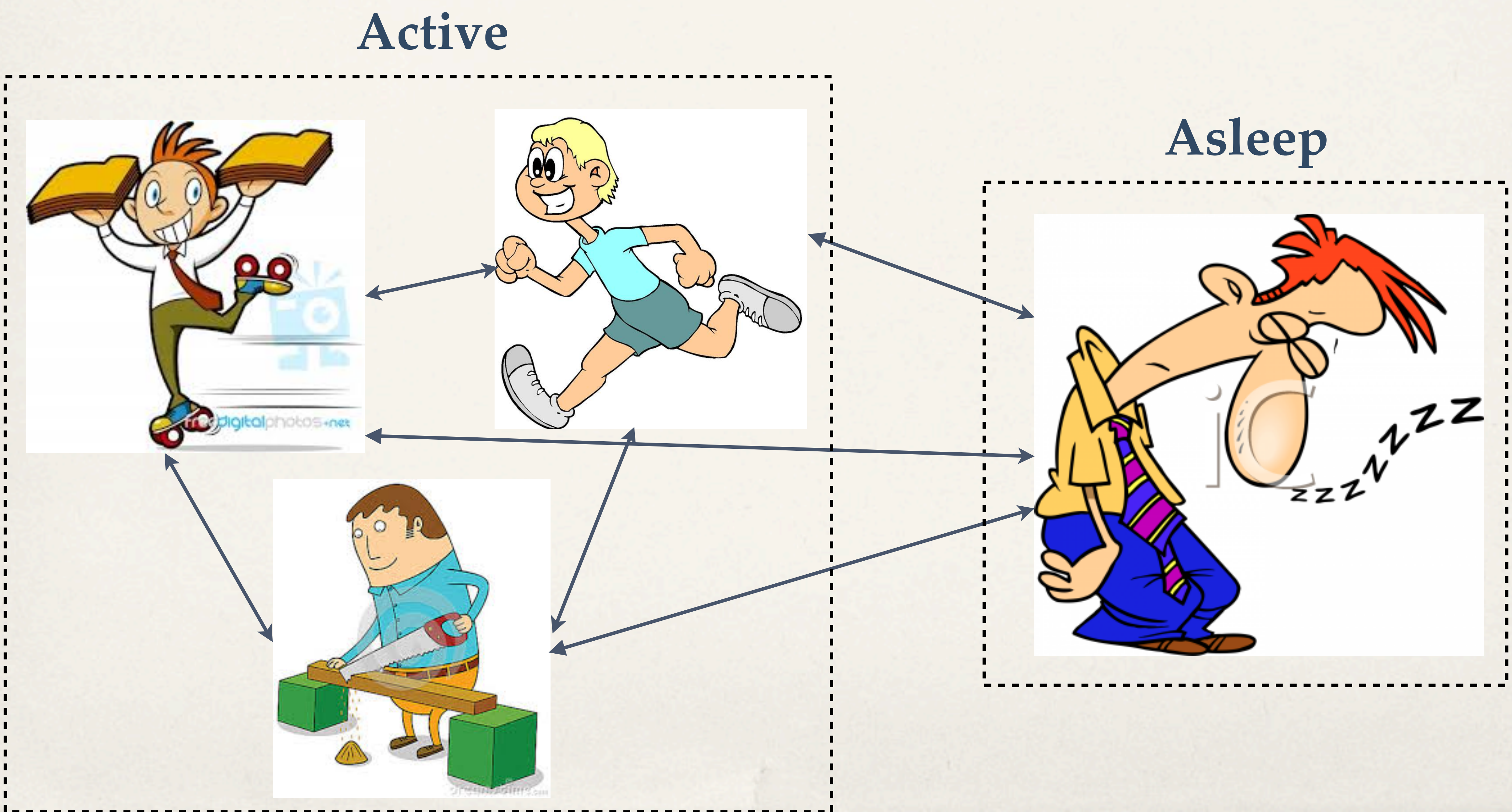
Motivation

- ε-machines are most useful when we have no understanding of the system-- perfect for biological modeling
- Problem: neurobiological data is highly subsampled.



Motivation

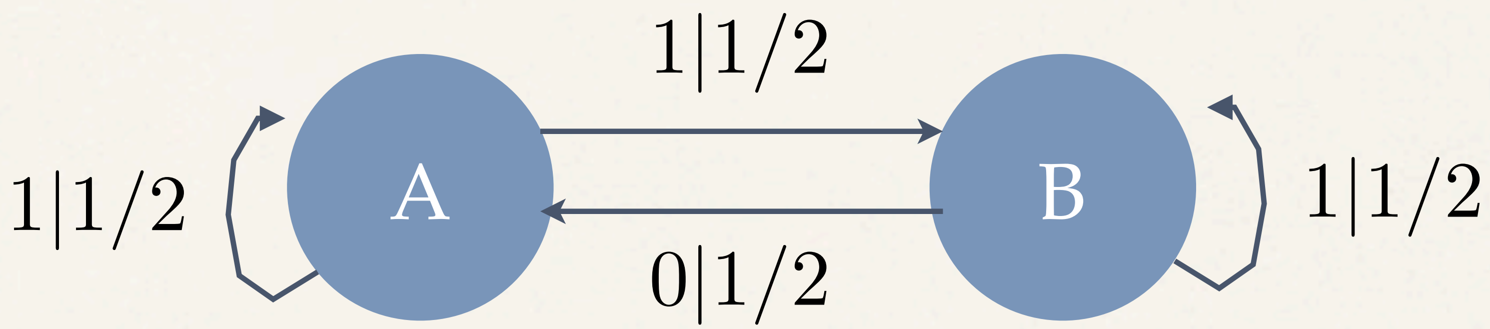
- These problems can maybe be couched as nonunifilar HMMs.



Outline: Results

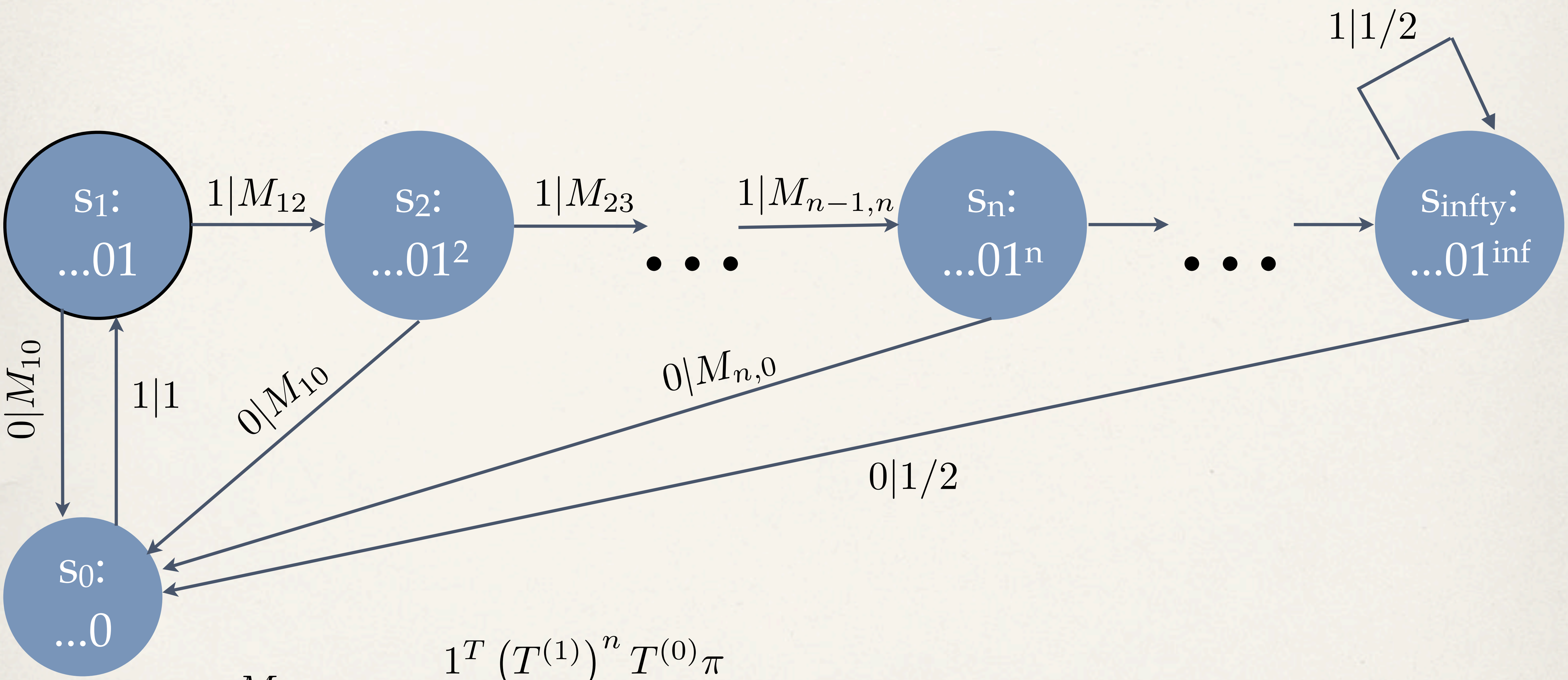
- Simple nonunifilar source (the one studied in class)
- Simple nonunifilar source with adjustable transition probabilities
    - Attempt to extend to continuous case
- Binary subsampled HMMs of a particular form, to be described

SNS



$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

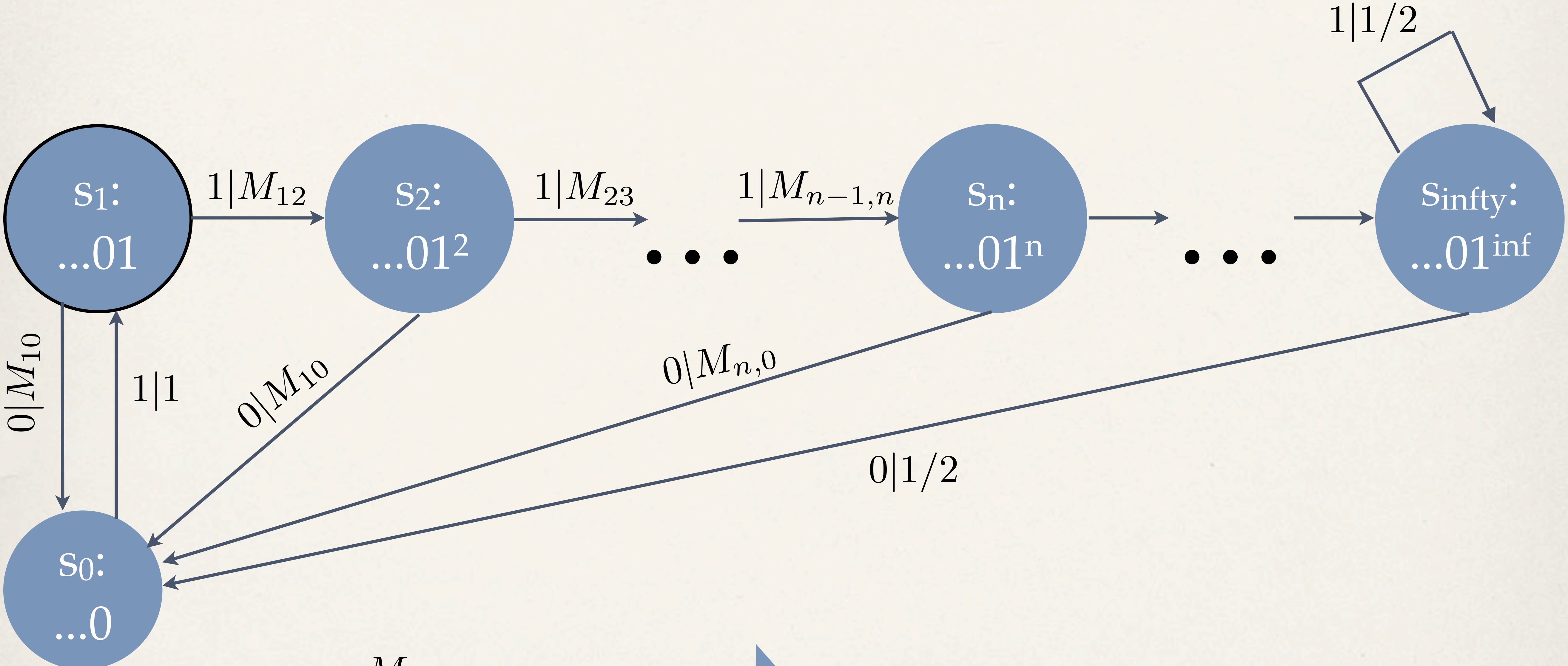
SNS



$$M_{n-1,n} = \frac{1^T \left(T^{(1)}\right)^n T^{(0)}\pi}{1^T \left(T^{(1)}\right)^{n-1} T^{(0)}\pi}$$

$$M_{n,0} = 1 - M_{n,n+1}$$

SNS



$$\pi_n = M_{n-1,n}\ \pi_{n-1}$$

$$\sum_{n=0}^{\infty} \pi_n = 1$$

$$\pi_n = \frac{1}{4}\frac{n+1}{2^n}$$

SNS: Stat. complexity and entropy rate

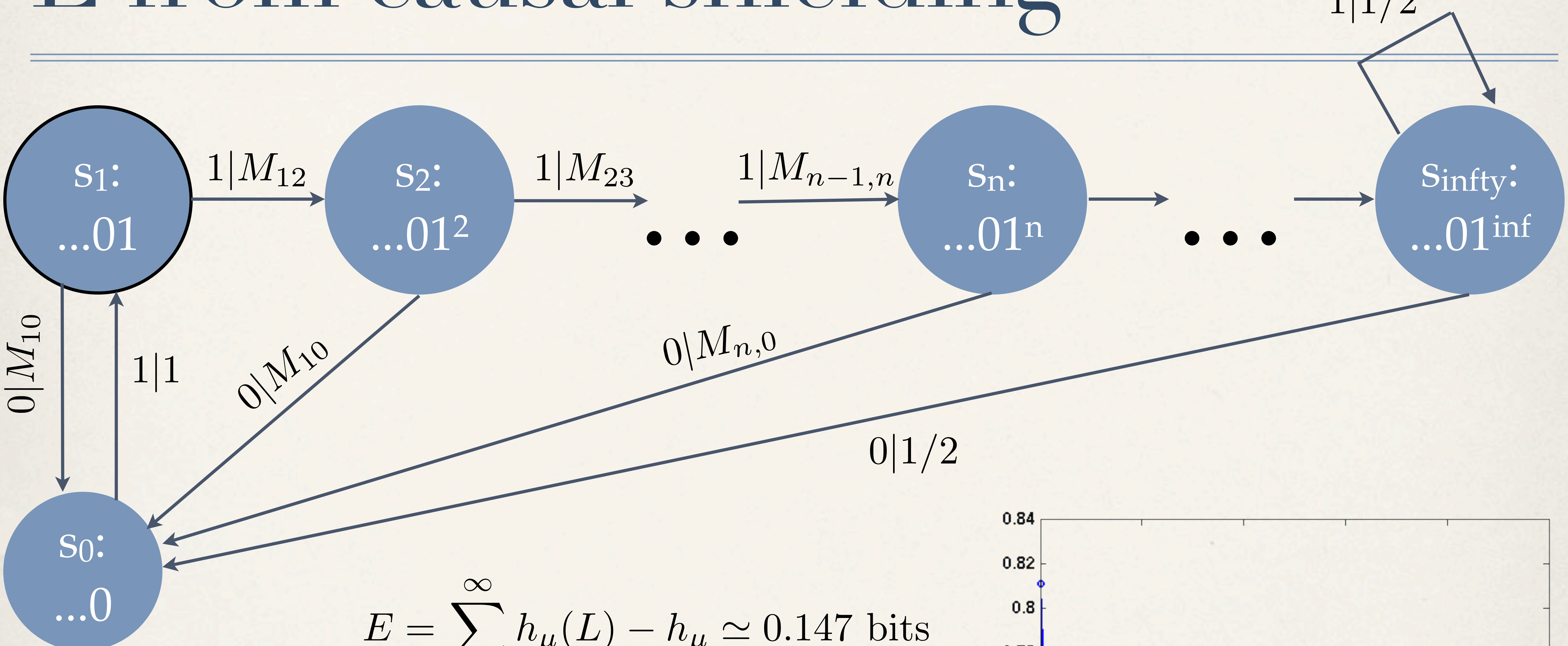
$$\pi_n = \frac{1}{4}\frac{n+1}{2^n}$$

$$M_{n-1,n} = \frac{n+1}{2n}$$

$$C_\mu = -\sum_{n=0}^{\infty} \pi_n \log_2 \pi_n = 2.71 \text{ bits}$$

$$h_\mu = \sum_{n=0}^{\infty} \pi_n H[M_{n,0}] = 0.678 \text{ bits}$$

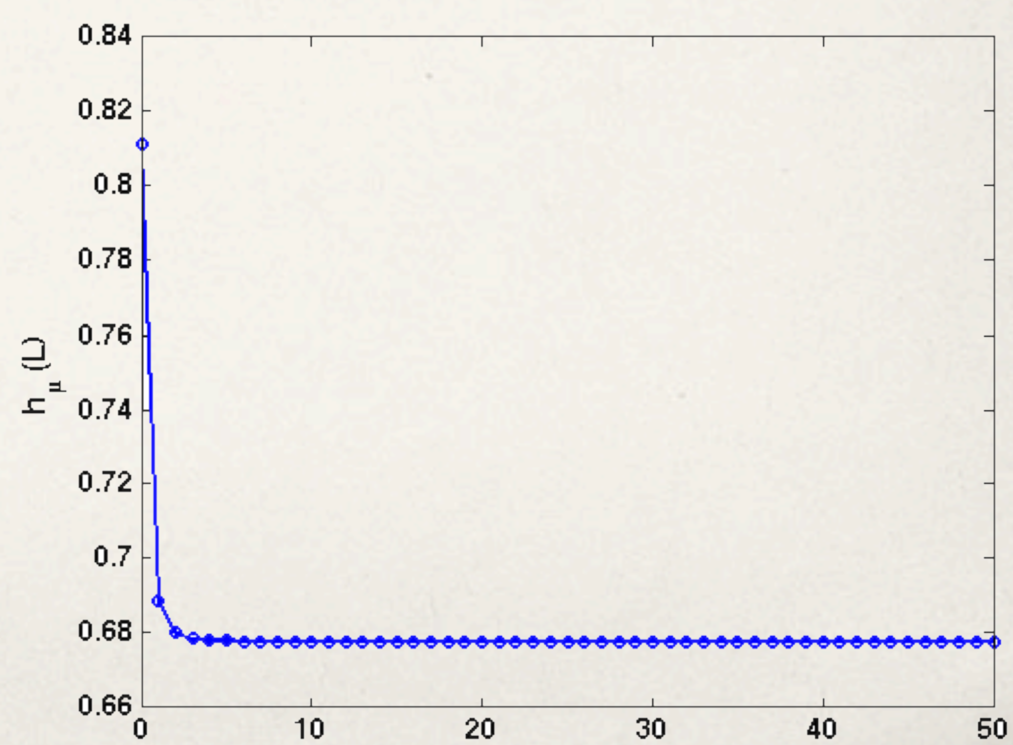
SNS:
E from causal shielding



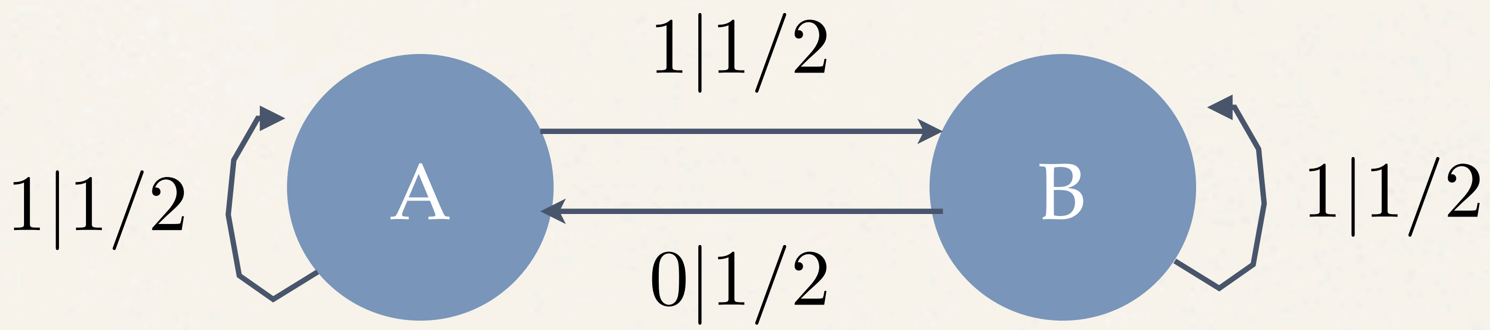
$$E = \sum_{L=0}^{\infty} h_\mu(L) - h_\mu \simeq 0.147 \text{ bits}$$

$$h_\mu(L) = H(L+1) - H(L) = H[X_{L+1}|R_{L+1}, R_0 = \mu_0]$$

$$\chi = C_\mu - E = 2.56 \text{ bits}$$



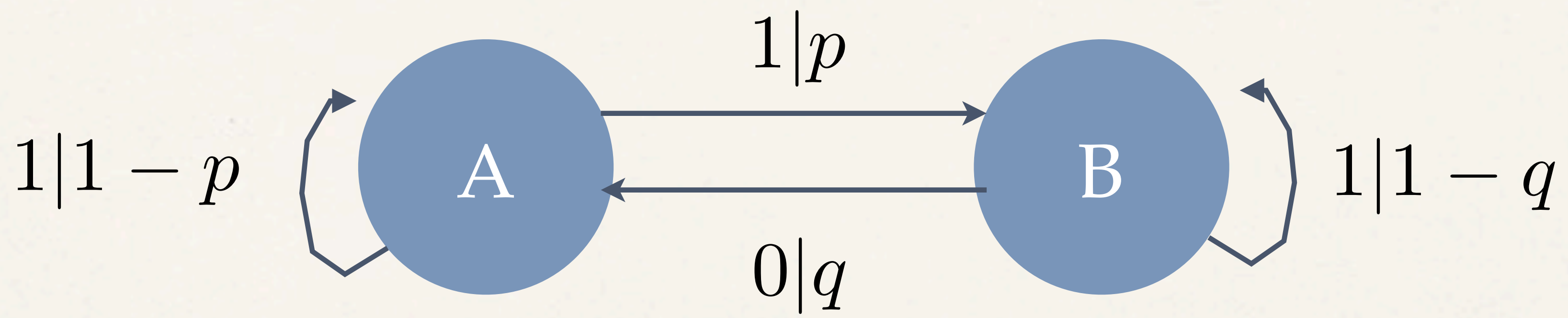
SNS:
Time reversed process?



$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

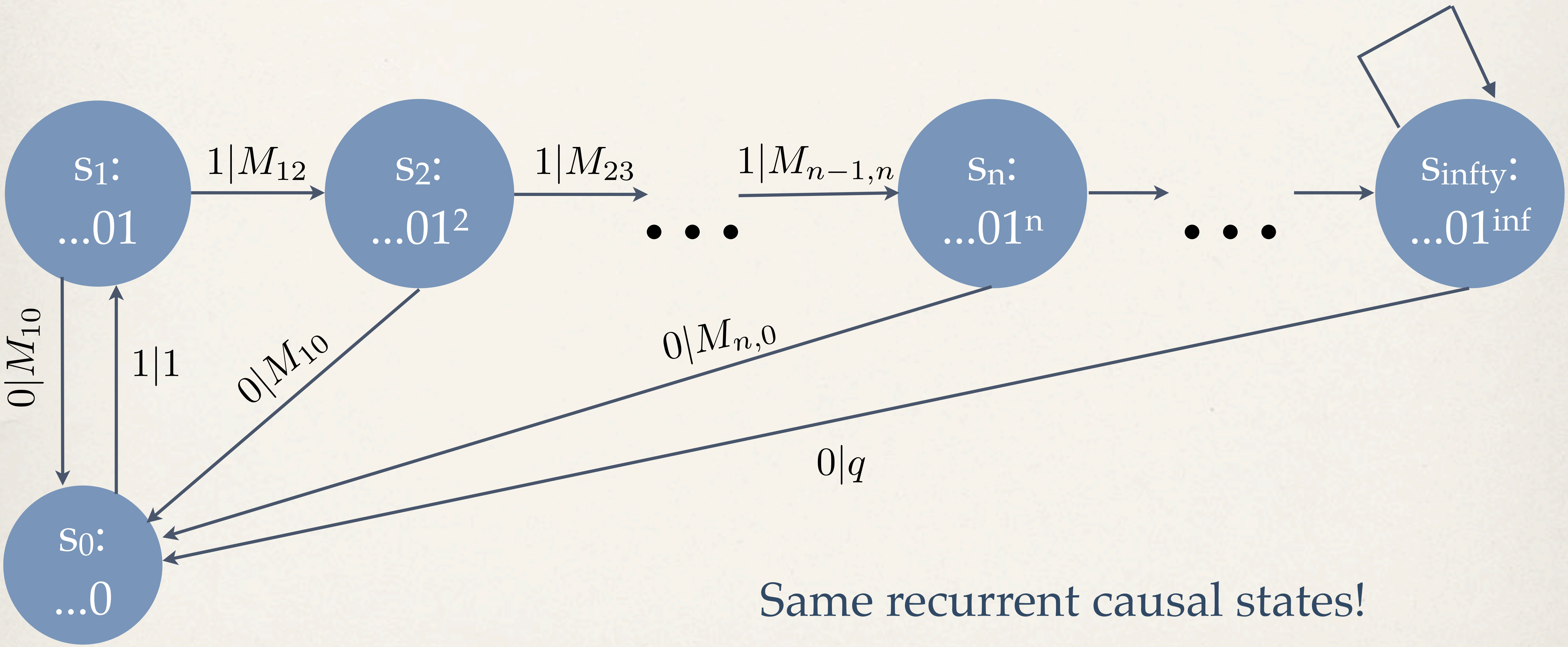
$$C_\mu^+ = C_\mu^-, \ \chi^+ = \chi^-, \ \Xi = 0$$

SNS v. 2



$$T^{(1)} = \begin{pmatrix} 1-p & 0 \\ p & 1-q \end{pmatrix}, \; T^{(0)} = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}, \; \pi = \begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix}$$

SNS v. 2

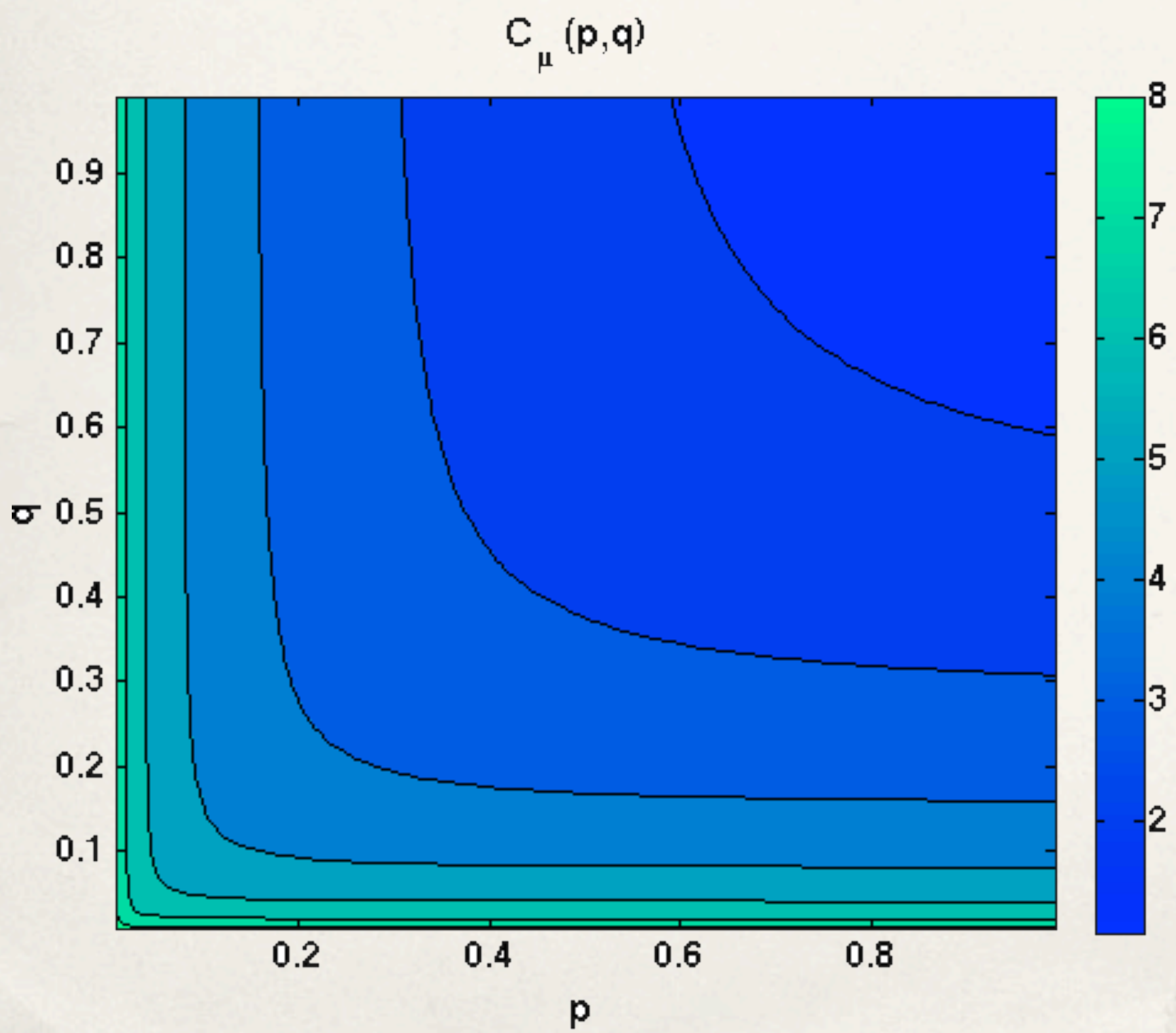


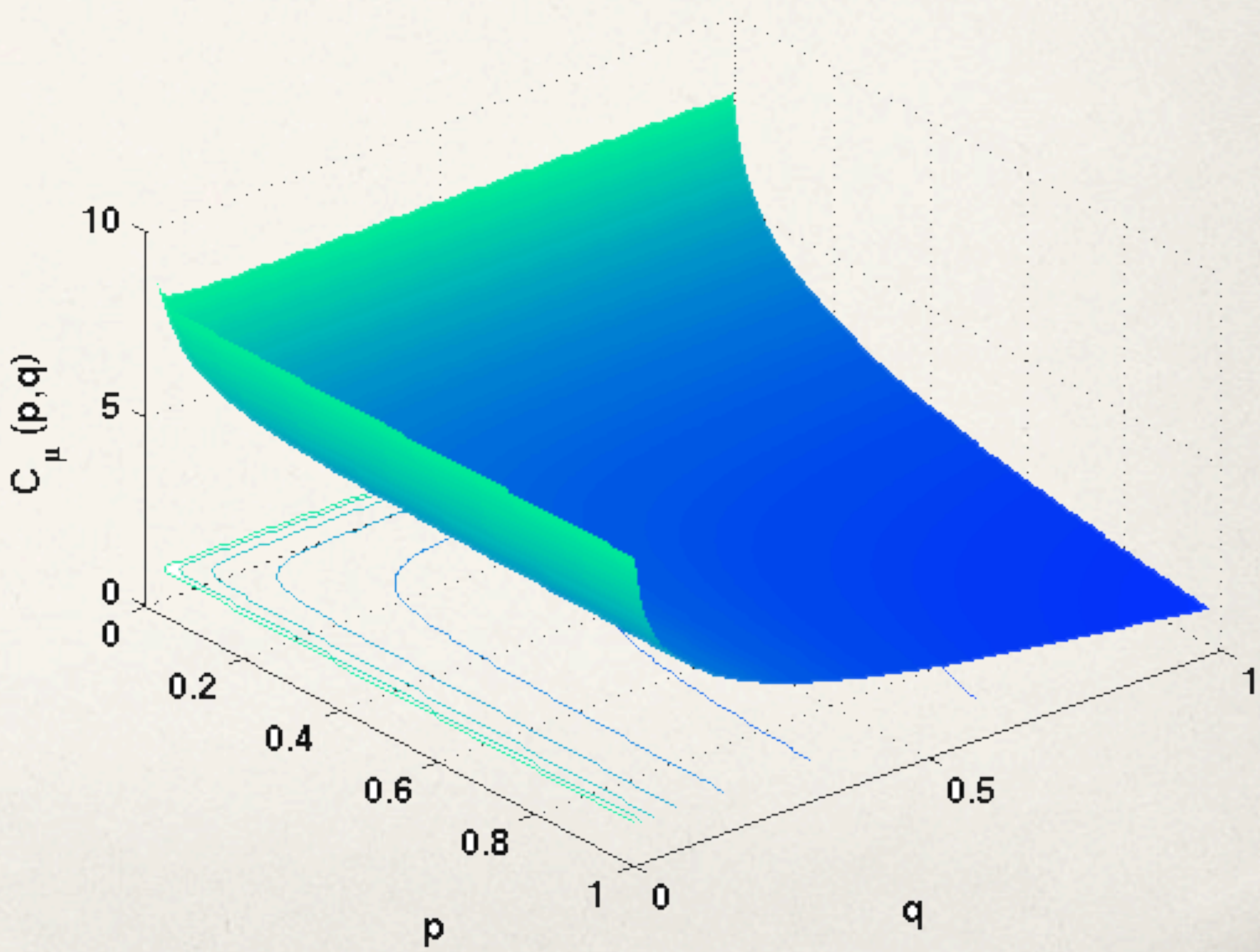
Same recurrent causal states!

$$M_{n-1,n} = \frac{1^T \left(T^{(1)}\right)^n T^{(0)} \pi}{1^T \left(T^{(1)}\right)^{n-1} T^{(0)} \pi} = \frac{p(1-q)^n - (1-p)^n q}{p(1-q)^{n-1} - q(1-p)^{n-1}}$$

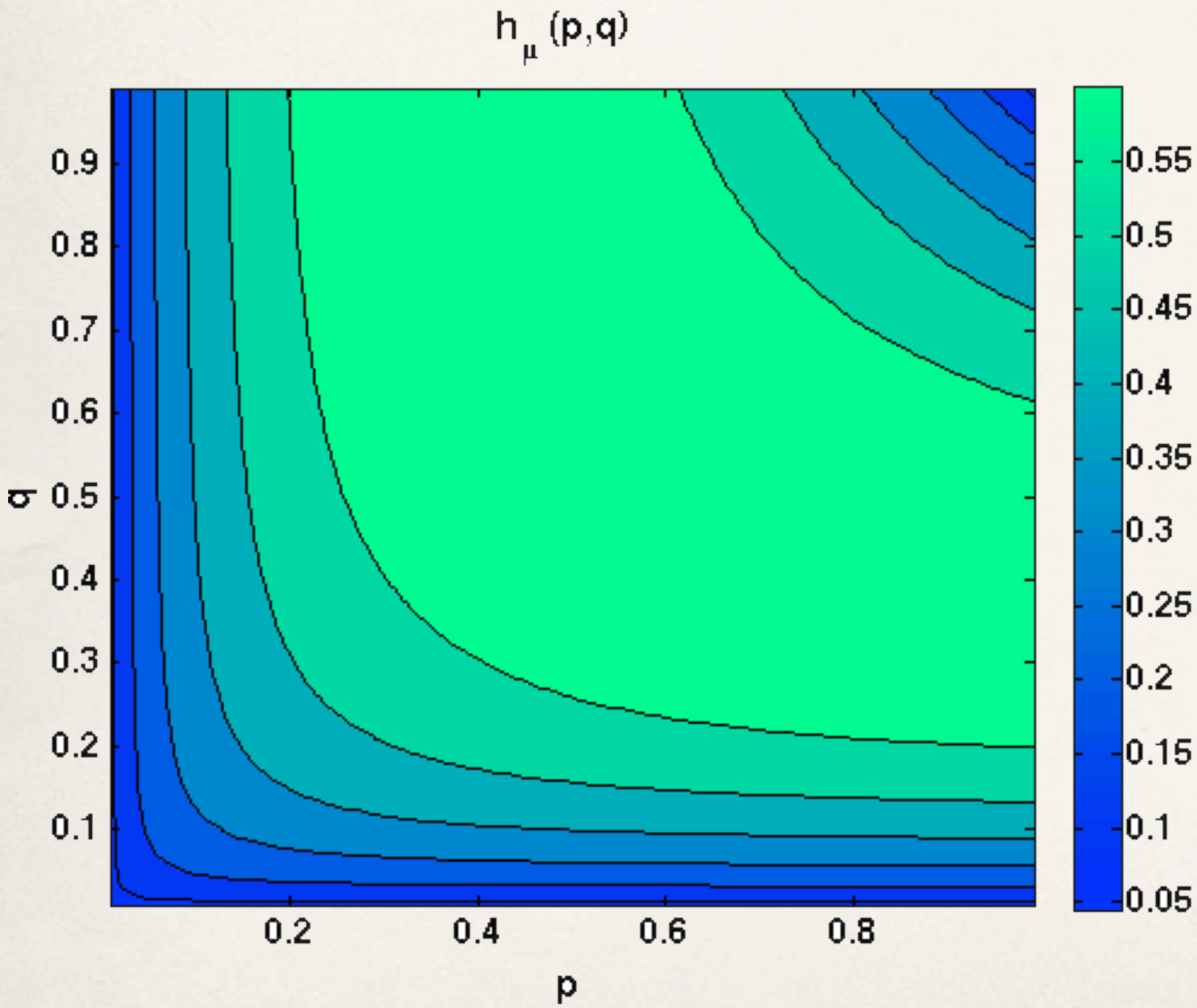
SNS v. 2

$$\pi_n = \frac{p(1-q)^n - q(1-p)^n}{p-q} \times \frac{pq}{p+q}$$

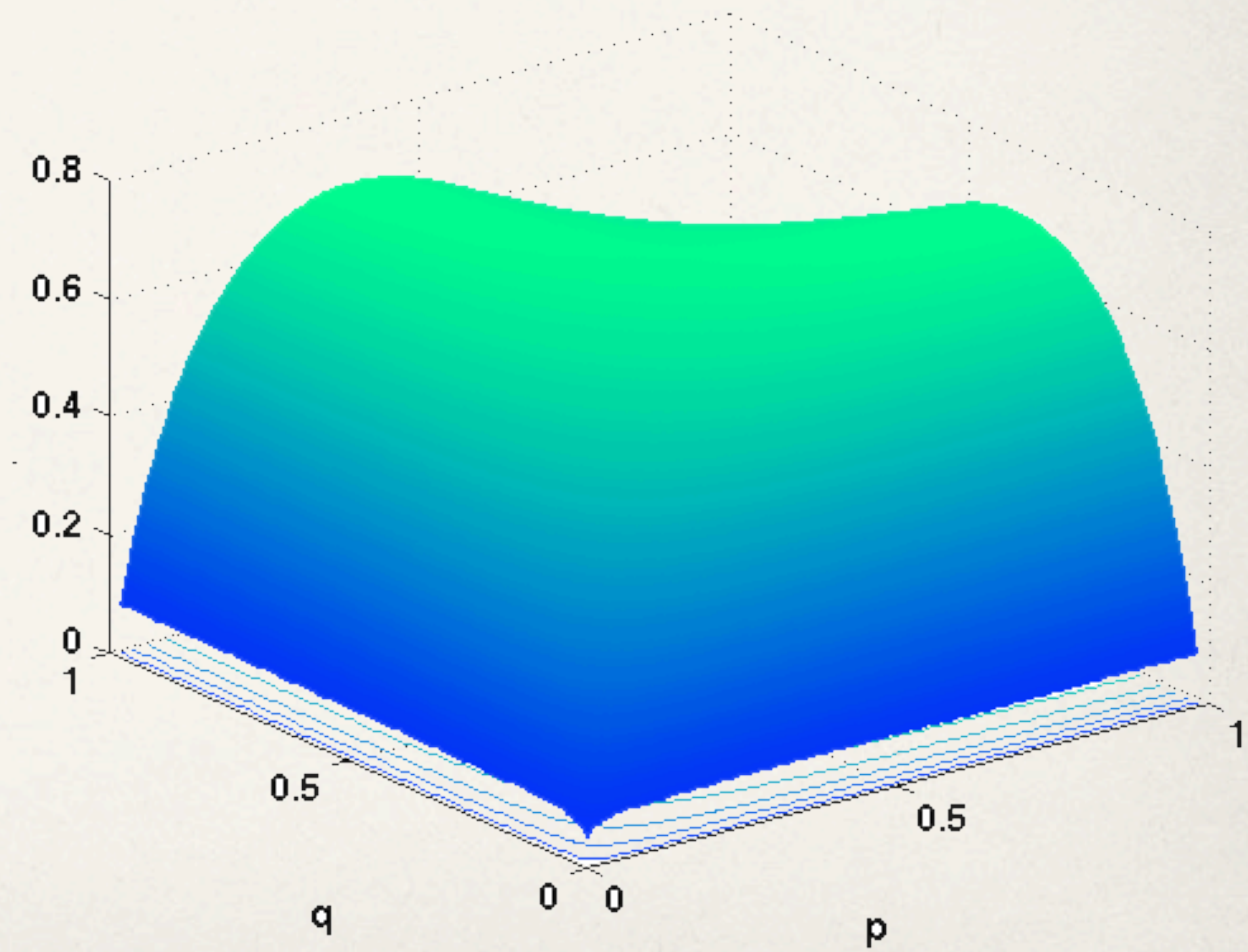




SNS v. 2

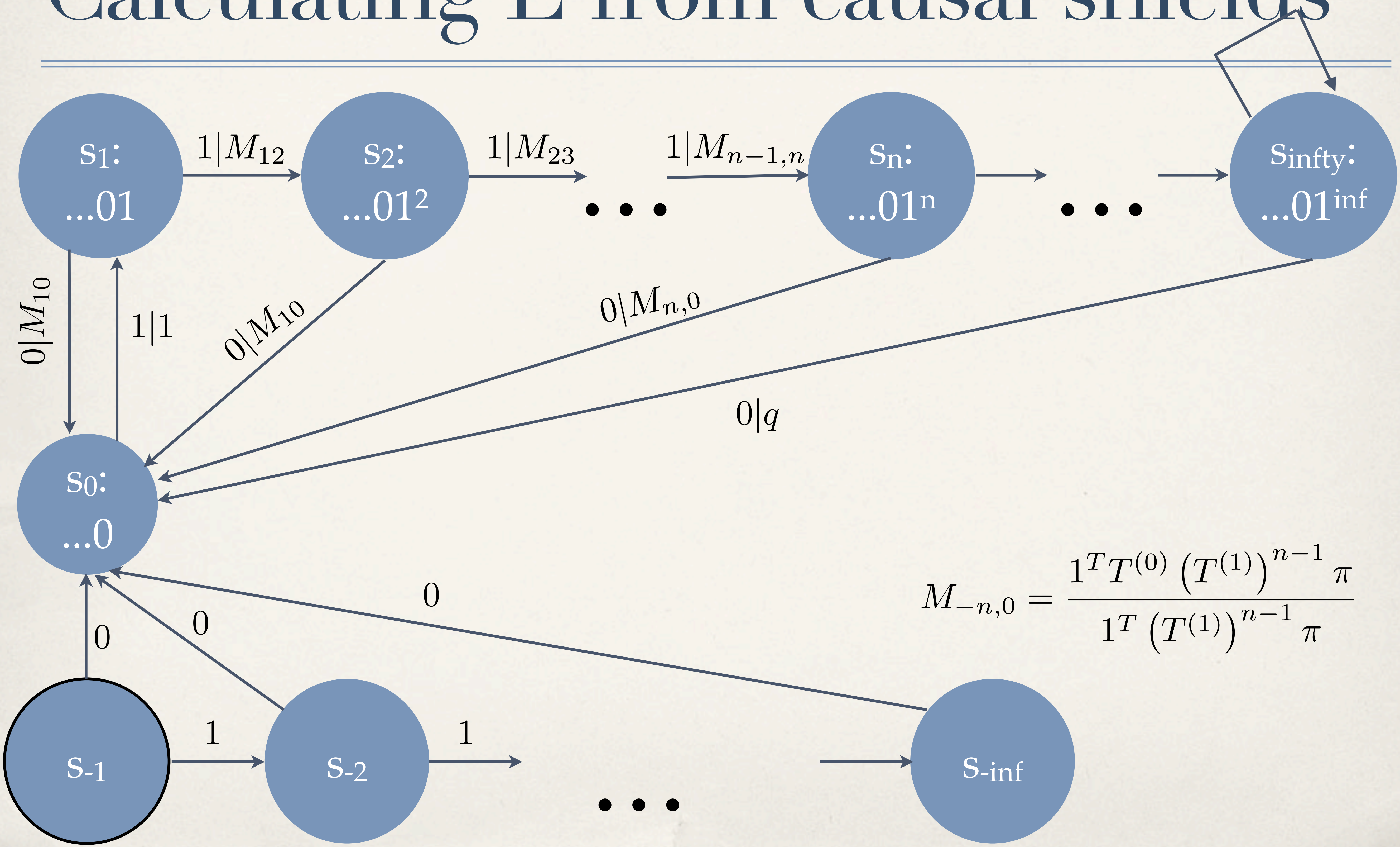


$h_\mu(p,q)$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
q
0.2
0.4
0.6
0.8
p
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05



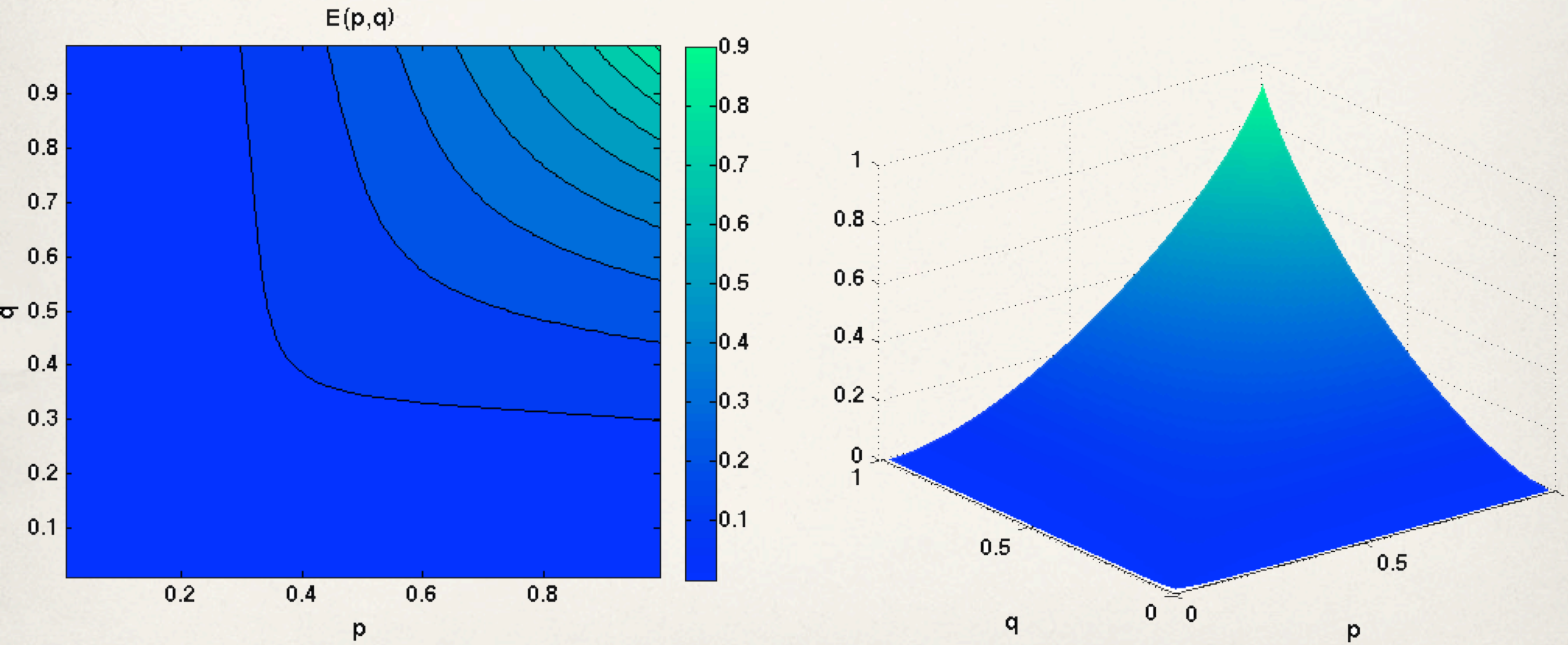
0.8
0.6
0.4
0.2
0
1
0.5
0
q
0
0.5
1
p

SNS v. 2: Calculating E from causal shields



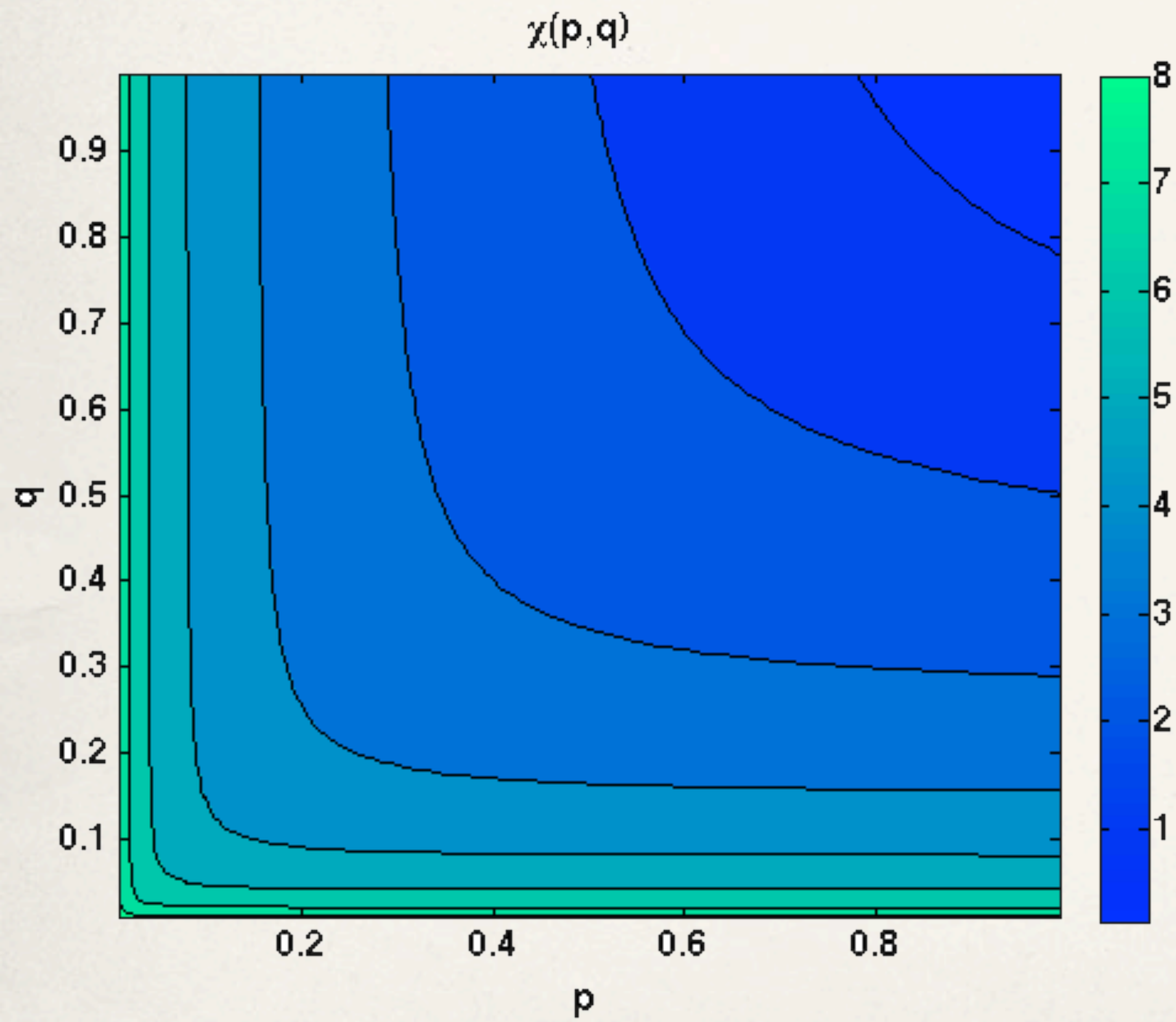
$$M_{-n,0} = \frac{1^T T^{(0)} \left(T^{(1)}\right)^{n-1} \pi}{1^T \left(T^{(1)}\right)^{n-1} \pi}$$

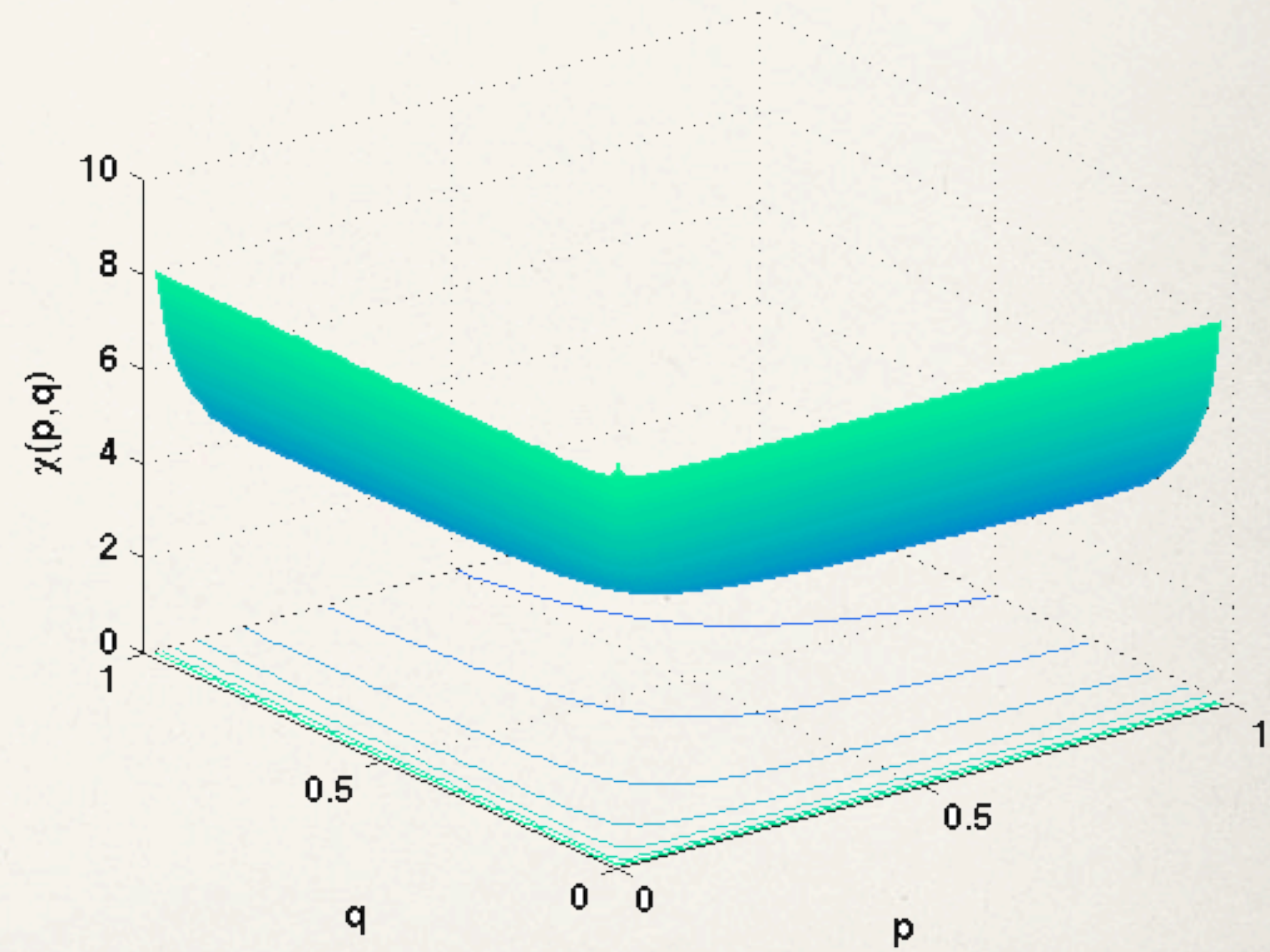
SNS v. 2



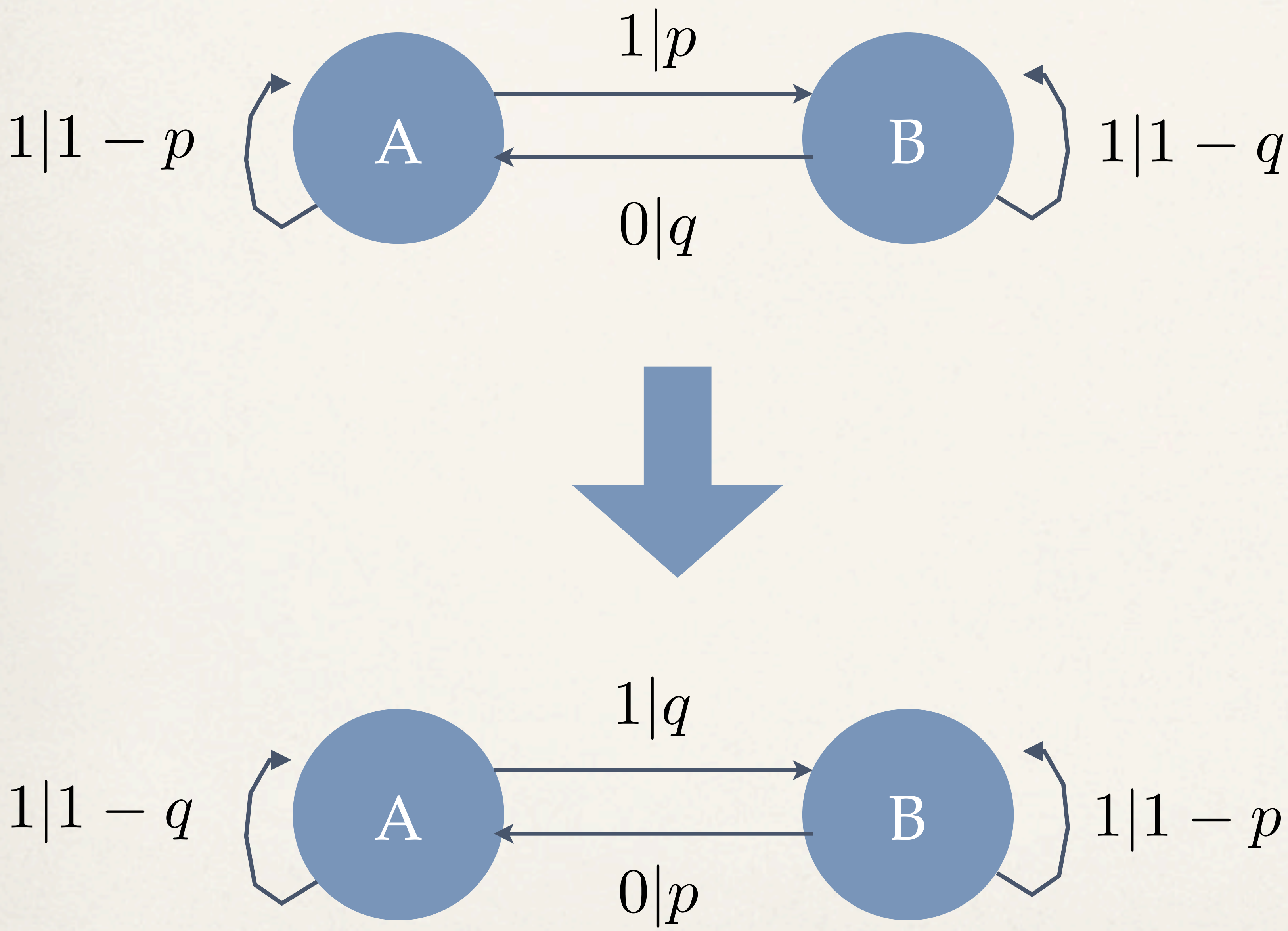
E(p,q)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
q
0.2
0.4
0.6
0.8
p
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1
0.8
0.6
0.4
0.2
0
1
0.5
0
q
0
0.5
1
p

SNS v. 2


χ(p,q)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
q
0.2
0.4
0.6
0.8
p
8
7
6
5
4
3
2
1


10
8
6
4
2
0
χ(p,q)
1
0.5
0
q
0
0.5
1
p

SNS v. 2: Time reversed process?



$$C_\mu(p,q) = C_\mu(p,q)$$
$$\Rightarrow C_\mu^+ = C_\mu^-$$
$$\Rightarrow \chi_\mu^+ = \chi_\mu^-$$
$$\Rightarrow \Xi = 0$$

SNS v. 2: Attempt at continuous time

Continuous time

$$\frac{d}{dt}\begin{pmatrix} p(A,t) \\ p(B,t) \end{pmatrix} = \begin{pmatrix} -k_{AB} & k_{BA} \\ k_{AB} & -k_{BA} \end{pmatrix} \begin{pmatrix} p(A,t) \\ p(B,t) \end{pmatrix}$$

Discretized time

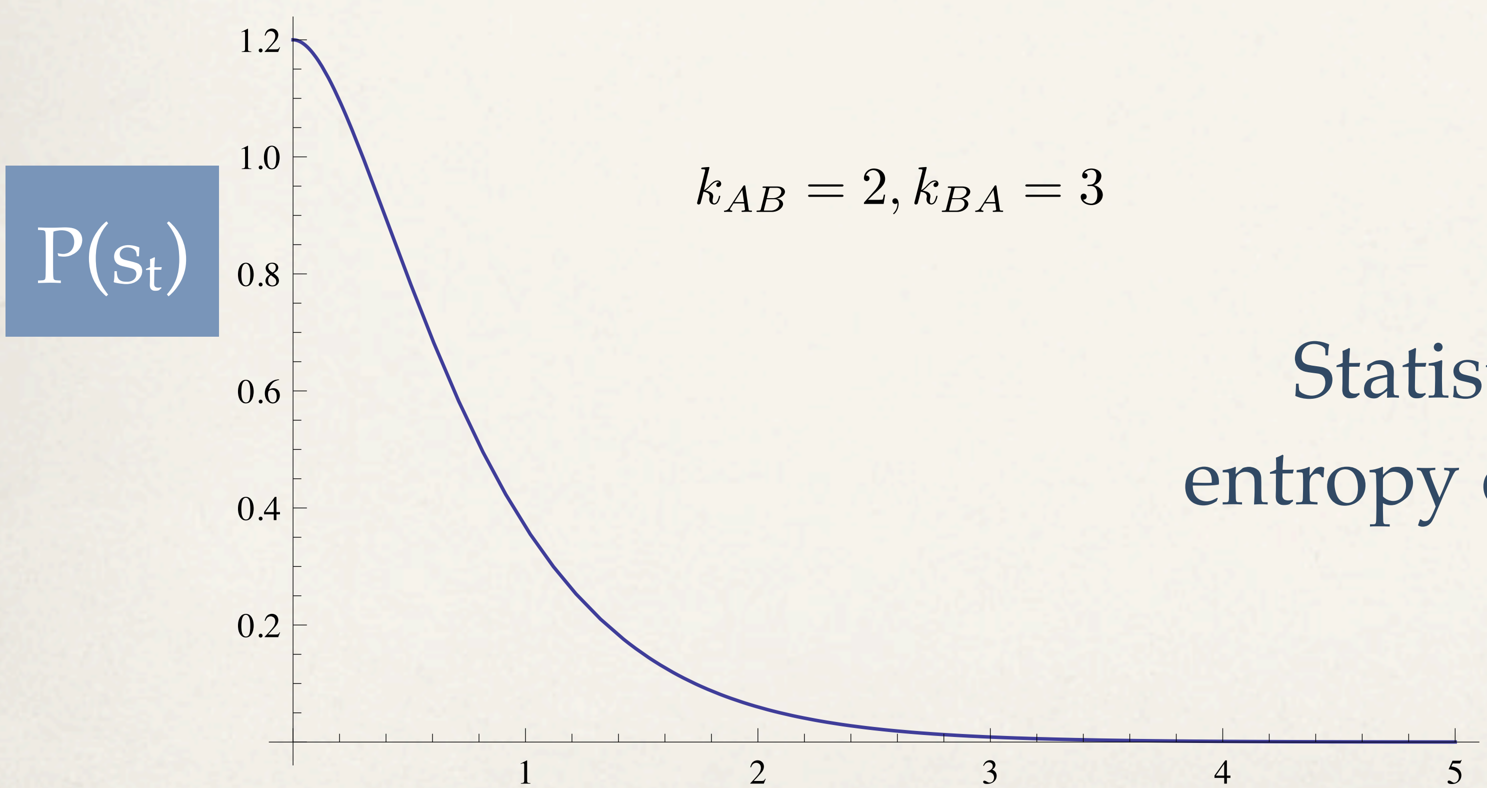
$$\begin{pmatrix} p(A,t+\Delta t) \\ p(B,t+\Delta t) \end{pmatrix} = \begin{pmatrix} 1-k_{AB}\Delta t & k_{BA}\Delta t \\ k_{AB}\Delta t & 1-k_{BA}\Delta t \end{pmatrix} \begin{pmatrix} p(A,t) \\ p(B,t) \end{pmatrix}$$

$$\Rightarrow p = k_{AB}\Delta t, \ \ q = k_{BA}\Delta t$$

SNS v. 2

$$\pi_t \Delta t = \lim_{\Delta t \to 0, n\Delta t = t} \pi_n(p = k_{AB}\Delta t, q = k_{BA}\Delta t)$$

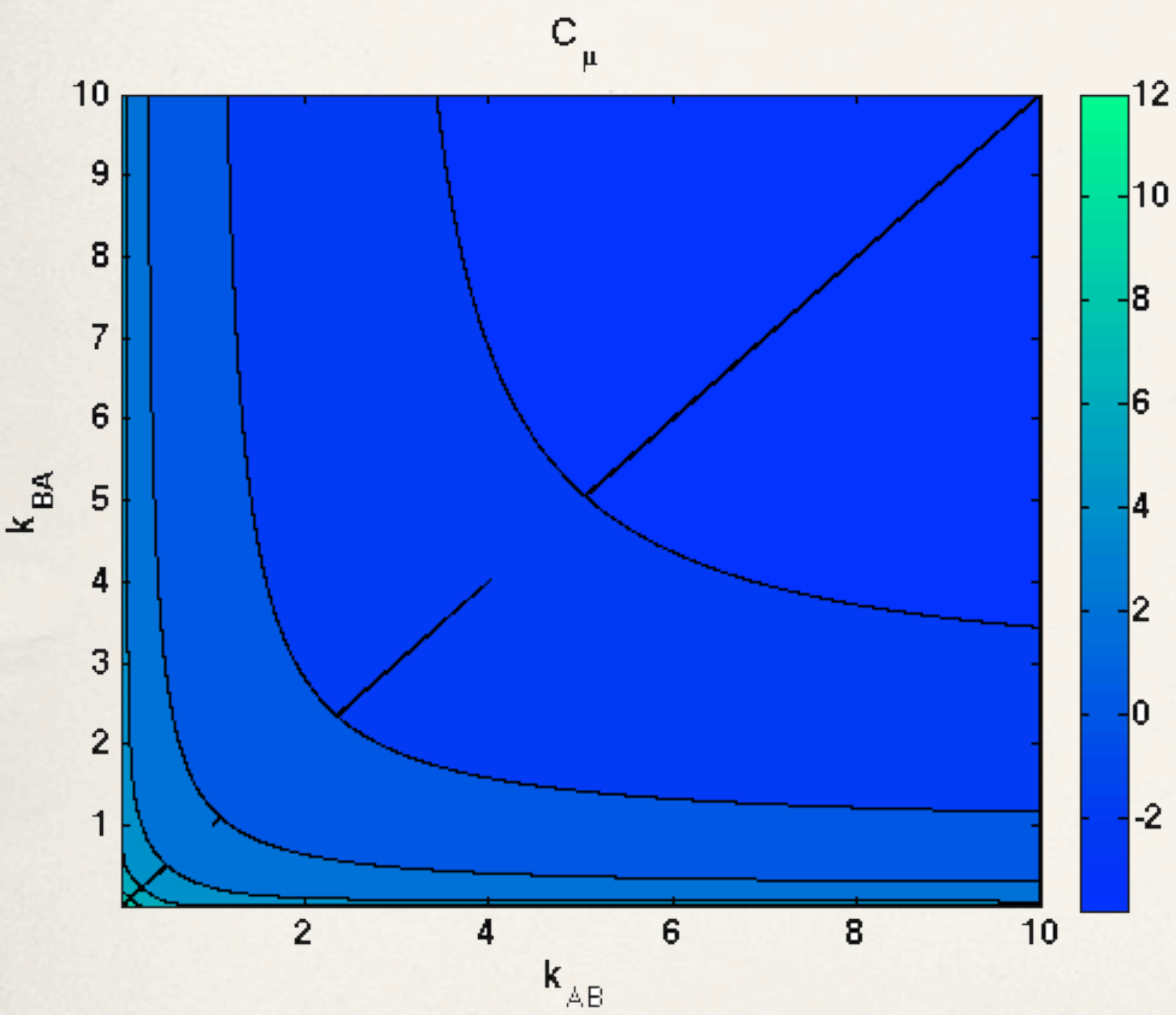
$$\pi_t = \frac{k_{AB}k_{BA}}{k_{AB} + k_{BA}} \frac{k_{AB}e^{-k_{BA}t} - k_{BA}e^{-k_{AB}t}}{k_{AB} - k_{BA}}$$

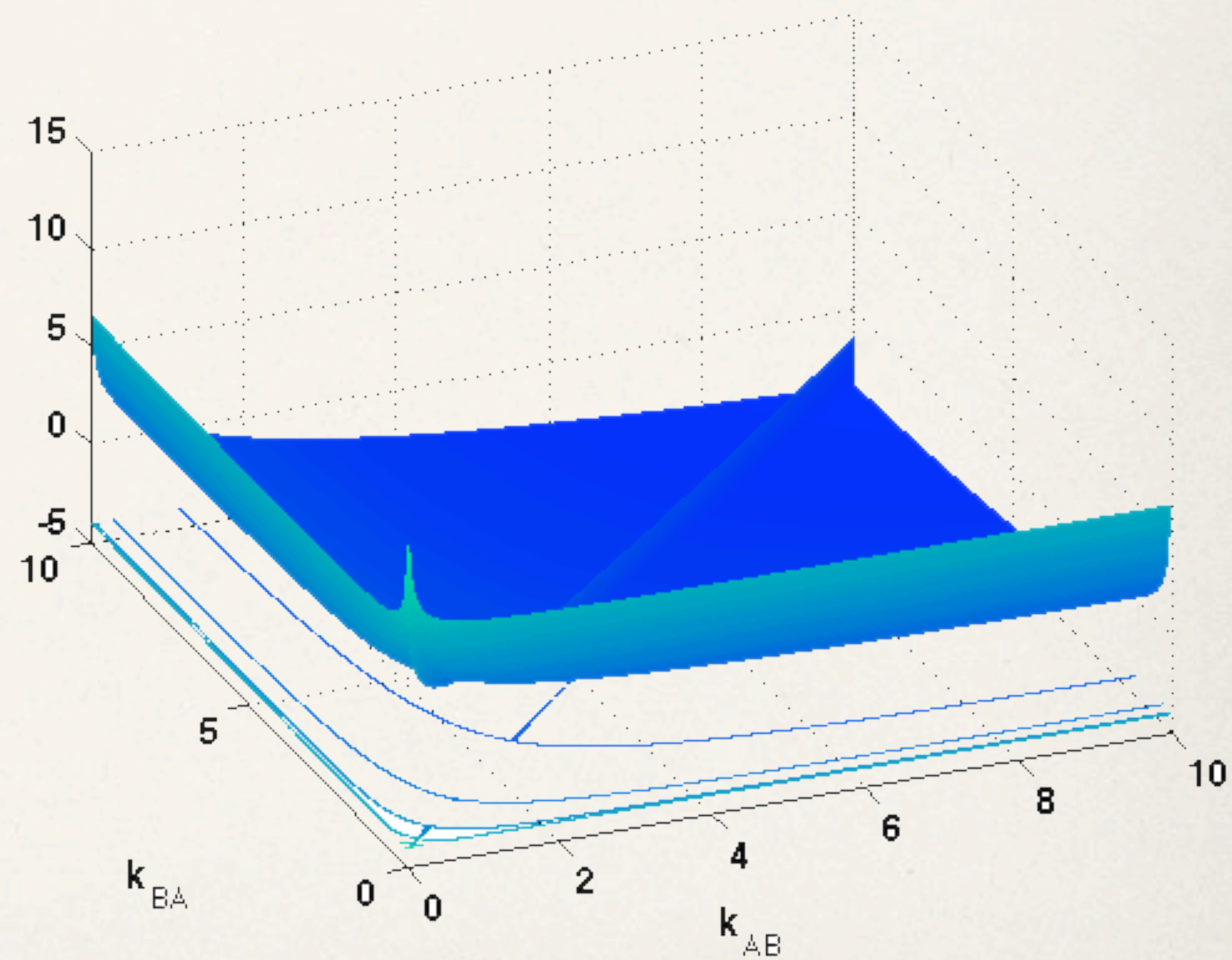


Statistical complexity: differential entropy of this probability distribution?

s_t

SNS v. 2: Continuous time stat. comp.





SNS v. 2

$$h_t = H[\frac{k_A k_B \left(k_A e^{-k_B t} - k_B e^{-k_A t}\right)}{k_A^2 e^{-k_B t} - k_B^2 e^{-k_A t}} \Delta t]$$

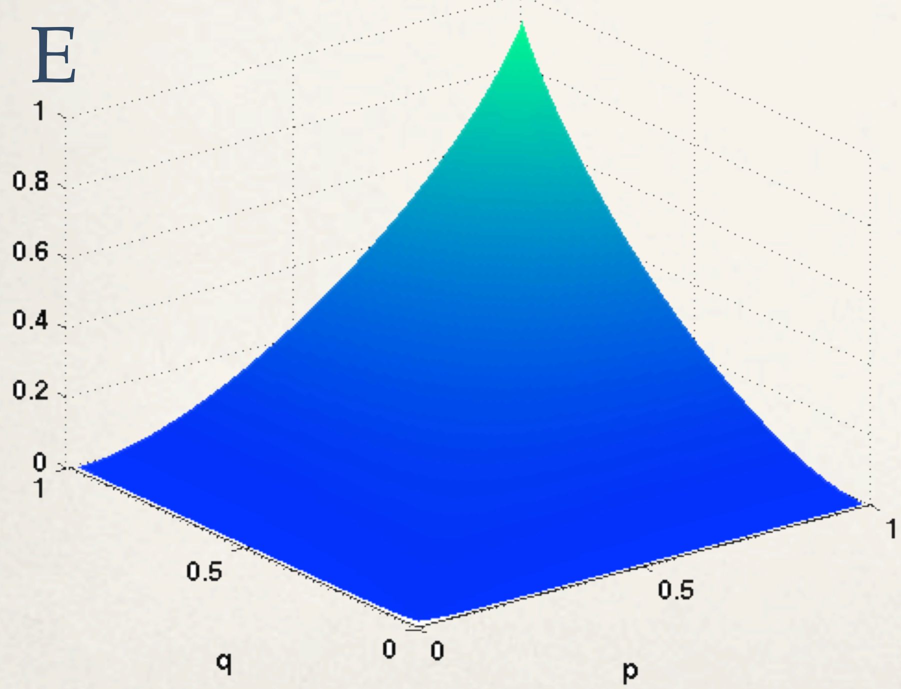
$$= \frac{k_A k_B \left(k_A e^{-k_B t} - k_B e^{-k_A t}\right)}{k_A^2 e^{-k_B t} - k_B^2 e^{-k_A t}} \Delta t \left(\frac{1}{\log 2} - \log_2 \frac{k_A k_B \left(k_A e^{-k_B t} - k_B e^{-k_A t}\right)}{k_A^2 e^{-k_B t} - k_B^2 e^{-k_A t}} \right)$$

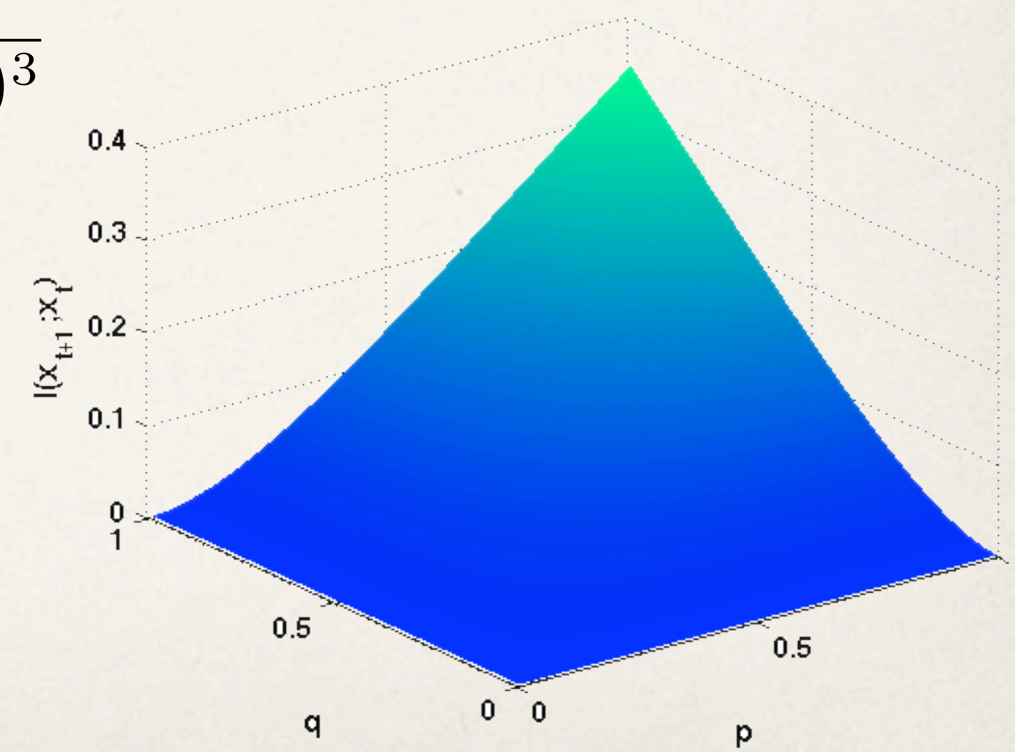
$$- \frac{k_A k_B \left(k_A e^{-k_B t} - k_B e^{-k_A t}\right)}{k_A^2 e^{-k_B t} - k_B^2 e^{-k_A t}} \Delta t \log_2 \Delta t$$

Not sure what to do with these weird factors of time resolution-- they seem to suggest the entropy rate is 0.

SNS v. 2: Excess entropy in cont. time?

- Did not unifilarize the time-reversed epsilon machine, so did not get a closed form analytic expression for excess entropy
- However, if excess entropy is mainly coming from the rule "a 0 must be followed by a 1" then $E \sim \frac{k_{AB}^2 k_{BA}^2 \Delta t}{(k_{AB}+k_{BA})^3}$

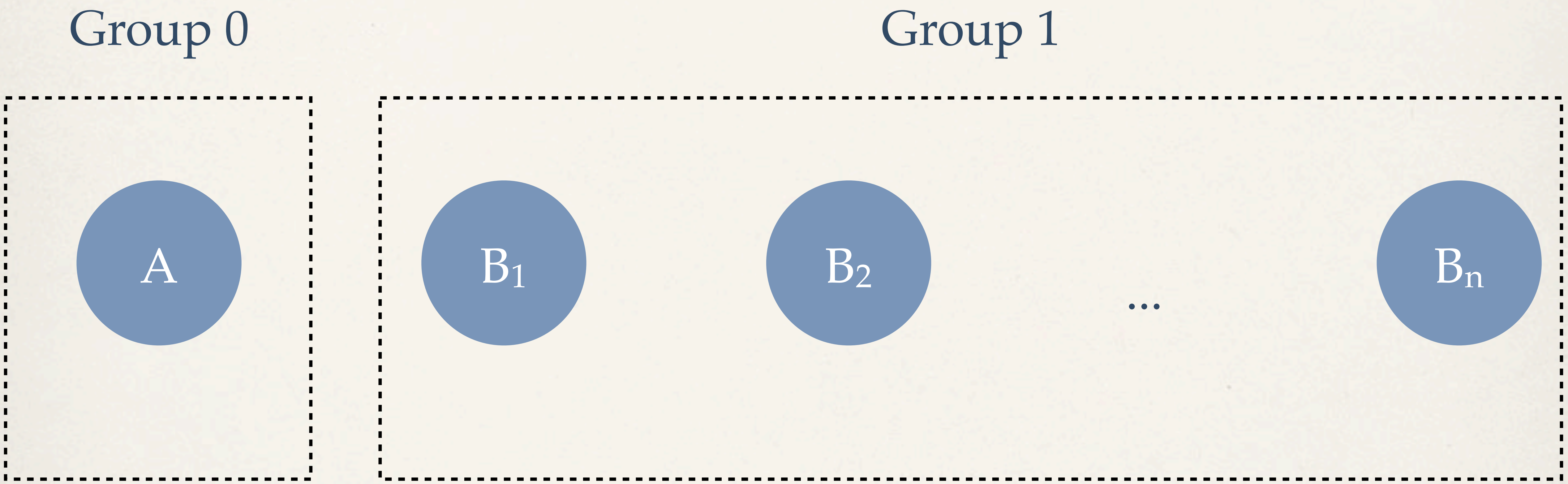




SNS v. 2

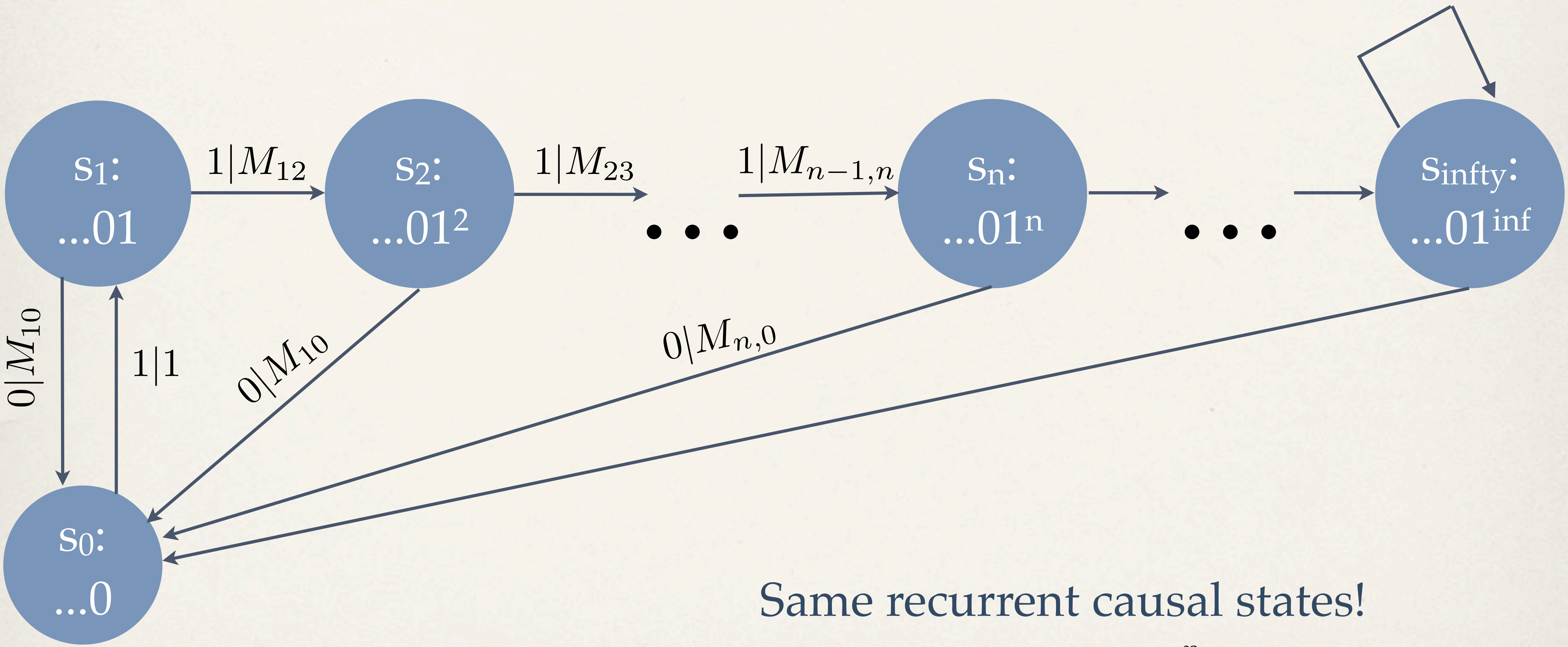
- Excess entropy and statistical complexity capture very different ideas.
- E captures how often you are synchronized to internal states
- Stat. comp. captures how long-tailed the probability distribution over causal states is
- Going to continuous time maybe introduces an uncountable infinity of causal states, differential entropies (negative stat. comp.???), discontinuities in stat. comp. vs. parameters
- E captures relaxation of probability distribution over *all* mixed states to stationarity

Last nonunifilar model



Fully connected, randomly chosen
kinetic rates between states

Last nonunifilar model



Same recurrent causal states!

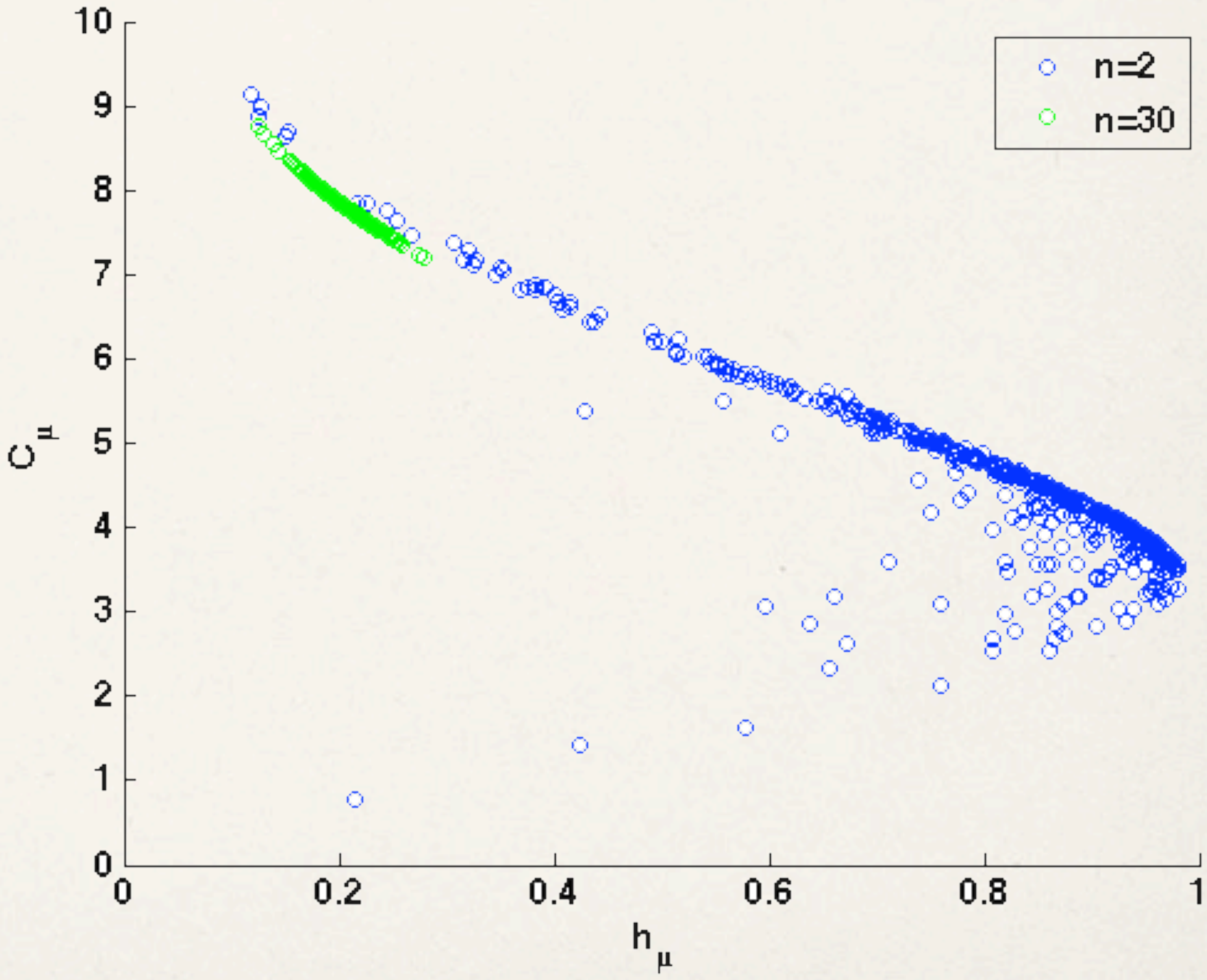
$$M_{n-1,n} = \frac{1^T \left(T^{(1)}\right)^n T^{(0)} \pi}{1^T \left(T^{(1)}\right)^{n-1} T^{(0)} \pi}$$

Preliminary results

This n is # of hidden states

$$\pi_n = \frac{1^T \left(T^{(1)}\right)^n T^{(0)}\pi}{1^T \left(I - T^{(1)}\right)^{-1} T^{(0)}\pi}$$

$$h_n = H[\frac{1^T \left(T^{(1)}\right)^{n+1} T^{(0)}\pi}{1^T \left(T^{(1)}\right)^n T^{(0)}\pi}]$$



Future directions

- Finish up calculating stuff for the last nonunifilar model.
- Maybe this has a practical application-- you can estimate the number of hidden states by knowing the average transition rates and calculating crypticity? We'll see.
- More nonunifilar models, continuous time, everything.